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A focal index theorem for null geodesics

PAUL E. EHRLICH (*)
University of Florida
Department of Mathematics
201 Walker Hall
Gainesville, FL 32611 U.S.A.

SEON-BU KIM (**)
Chonnam National University
Department of Mathematics
Kwangju City
500 Korea, South Korea

Abstract. *Employing techniques recently developed by D. Kalish for Riemannian manifolds, we obtain a focal Morse index theorem for a null geodesic segment initially and terminally perpendicular to spacelike submanifolds of arbitrary codimension in a general space-time.*

SECTION 1: INTRODUCTION

Let $\beta : [0,1] \to (M,g)$ be a null geodesic segment in an arbitrary space-time of dimension $n \geq 3$ and let K_1, K_2 be spacelike submanifolds perpendicular to β at $\beta(0)$ and $\beta(1)$, respectively. The purpose of this paper is to extend the conjugate

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point Morse index theorem for a null geodesic segment $\beta : [0,1] \to (M,g)$ given in Beem and Ehrlich [3], cf. [4, Section 9.3] to the focal case of variations perpendicular to K_1 and K_2, assuming that $t = 1$ is not a K_1-focal point to $t = 0$ along β.

To gain a perspective on the focal index theory, we review certain aspects of the conjugate point Morse theory for a unit speed geodesic $c : [a,b] \to (N, g_0)$ in a Riemannian manifold (N, g_0). In the conjugate point case, one considers the vector space $V_0^\perp(c)$ of continuous, piecewise smooth vector fields Y along c perpendicular to c with boundary conditions $Y(a) = Y(b) = 0$ and the associated index form $I : V_0^\perp(c) \times V_0^\perp(c) \to \mathbb{R}$ arising from the second variation of arc length and given by

$$I(X,Y) = \int_{t=a}^{b} (g_0(X',Y') - g_0(R(X,c')c',Y))\,\mathrm{d}t. \tag{1.1}$$

The index form (1.1) is negative semi-definite up to and including the first conjugate point to $t = a$ along c, so that it is sensible to define the index $\mathrm{Ind}(c)$ of c by

$$\mathrm{Ind}(c) = \sup\Big\{\dim A;\ A \text{ is an } \mathbb{R} \text{ vector subspace of } V_0^\perp(c) \text{ on which } I\,|_{A\times A} \text{ is positive definite}\Big\}.$$

Subdividing the geodesic c into a finite number of segments, each contained within a geodesically convex neighborhood, and approximating piecewise smooth variational vector fields by piecewise smooth Jacobi vector fields, M. Morse obtained the important result tha $\mathrm{Ind}(c)$ is finite and is in fact given by summing the multiplicities of conjugate points to $t = a$ along c in (a,b), i.e.,

$$\mathrm{Ind}(c) = \sum_{t\in(a,b)} \dim J_t(c)$$

where $J_t(c)$ denotes the vector space of smooth Jacobi fields J along c with $J(a) = J(t) = 0$. Important in Morse's proof of the index theorem is the result that if c is restricted to $[a,t]$ for any $t \in [a,b]$, then the restricted index form

$$f(t) := \mathrm{Ind}\left(c\,|_{[a,t]}\right) \tag{1.2}$$

is nondecreasing as a function of t and is left continuous. (Also used is the fact that the augmented or extended index $f_0(t) = \mathrm{Ind}_0\left(c\,|_{[a,t]}\right)$ is nondecreasing and right continuous, and finally, $f_0(t) - f(t) = \dim J_t(c)$.) Geometrically, if a family $Y^{(t)}$ of piecewise smooth Jacobi fields along $c\,|_{[a,t]}$ with $Y^{(t)}(t) = 0$ is considered, then as $t \to t_0$, one has $I(Y^{(t)}, Y^{(t)}) \to I(Y^{(t_0)}, Y^{(t_0)})$.

In Beem and Ehrlich [4, chapter 9], it was shown that the proof method of Morse for the conjugate point Riemannian index theory may be extended to a timelike geodesic segment or to a null geodesic segment in an arbitrary space-time. The key observation here was that the appropriate analogue of (1.2) defines a nondecreasing function with the same continuity properties as in the Riemannian case, provided that a quotient bundle construction suggested by General Relativity theory was employed in the null case.

The index theory for the variational problem with geodesic $c : [a,b] \to (N, g_0)$ perpendicular at $p = c(a)$ to a submanifold K_1 and at $q = c(b)$ to a submanifold K_2 of the Riemannian manifold (N, g_0) has been found to be more complicated using Morse's approach for two reasons. First, to generalize Morse's conjugate point approach and study the index of $c\,|_{[a,t]}$, a «geometrically meaningful» submanifold $K(t)$ perpendicular at t to c with $K(t) \to K_2$ as $t \to b$ needs to be constructed. Then $K_1, K(t)$-Jacobi fields need to be studied, rather than the simpler conjugate point condition $Y^{(t)}(t) \to 0$ as $t \to t_0$ for a fixed $t_0 \in [0,1]$. This was done by Ambrose [2] using the geodesic flow of the normal bundle to K_1. This construction leads, however, to technical difficulties at the K_1-focal points at which the submanifolds $K(t)$ may lose dimension and differentiability. For this purpose, Ambrose [2] introduced a boundary condition as a pair of operators $S_1, S_2 : (c'(t))^{\perp} \to (c'(t))^{\perp}$ satisfying certain properties valid even if t is a K_1-focal point. In the case that t is not a K_1-focal point, S_1 corresponds to the projection map of $(c'(t))^{\perp}$ onto $T_{c(t)}(K(t))$ and S_2 to the second fundamental form of $K(t)$ as a submanifold of (N, g_0).

A second complication in the focal point index theory is that the index function $f(t)$ defined by analogy to (1.2) is no longer so nicely behaved. In 1961, Ambrose [2] studied this problem by breaking up the index function as a sum of the orders of what Ambrose calls the K_1, K_2-conjugate points and a convexity term arising from the difference in second fundamental forms of K_1 and the $K(t)$'s. Using this approach, Ambrose showed that the index function was nondecreasing, but failed to have the left and right continuity properties of the index and augmented index in the simpler conjugate point problem. In 1977, Bolton [6] obtained an index theorem using subdivision arguments like Morse and Ambrose, but working with an index function which fails to be nondecreasing. In 1984, Kalish [10] gave a new proof of the Morse index theorem for a geodesic $c : [a,b] \to (N, g_0)$ in a Riemannian manifold with perpendicular endmanifolds K_1, K_2 provided that $t = b$ is not a K_1-focal point to $t = a$ along c. Kalish's proof avoids the use of subdivision arguments and hence avoids the need to define a family $K(t)$ of submanifolds perpendicular to $c(t)$ as in [2], [6].

It is this third approach of Kalish [10] which we show in the present paper may be adapted to the case of a null geodesic segment $\beta : [0,1] \to (M, g)$ in an arbitrary Lorentzian manifold (M, g) with spacelike endmanifolds K_1, K_2 orthogonal at $\beta(0), \beta(1)$ respectively to β, again supposing that $t = 1$ is not a K_1-focal point to $t = 0$ along β. The proof entails adopting Kalish's arguments to the null quotient bundle setting and to K_1-vector classes. Similar results hold for arbitrary timelike geodesic segments with spacelike endmanifolds in general space-times.

SECTION 2: PRELIMINARIES

Let (M, g) be a connected Lorentzian manifold of dimension $n \geq 3$. Thus M is a smooth manifold with a countable basis and a smooth Lorentzian metric g of signature $(-,+,\ldots,+)$. Let D denote the Levi-Civita connection determined by g. A nonzero tangent vector $v \in TM$ is said to be timelike (resp. nonspacelike, null, spacelike) if $g(v,v) < 0$ (resp. $\leq 0, = 0, > 0$). An immersed smooth submanifold $f : K \to (M, g)$ is said to be spacelike if the pull back metric f^*g for K is positive definite. As usual, we will identify K and $f(K)$ and thus assume K is contained in M.

Let $\beta : [0, b] \to (M, g)$ be a fixed null geodesic segment and let K be a spacelike submanifold of dimension k of (M, g) with $\beta'(0)$ perpendicular to K at $p := \beta(0)$. From this perpendicularity, $k \leq n - 2$. We denote by $V^{\perp}(\beta)$ the $\mathbf{R}$-vector space of continuous, piecewise smooth vector fields Y along β with $g(Y, \beta') = 0$ and $V^{\perp}(\beta, K) := \{Y \in V^{\perp}(\beta); Y(0) \in T_pK\}, V_0^{\perp}(\beta, K) = \{Y \in V^{\perp}(\beta, K); Y(b) = 0\}$. Recall that a Jacobi field J along β is a smooth vector field satisfying the differential equation $J'' + R(J, \beta')\beta' = 0$. We will adopt the sign convention on the second fundamental tensor $S_{\beta'(0)} : T_pK \to T_pK$ that $S_{\beta'(0)}x := -(D_xN)^T$ where $T : T_pM \to T_pK$ denotes the projection map and N is any local vector field normal to K with $N(p) = \beta'(0)$. With this sign convention, we make the usual

DEFINITION 2.1. A (smooth) Jacobi field $J \in V^{\perp}(\beta, K)$ is a *K-Jacobi field* if $J(0) \in T_pK$ and $J'(0) + S_{\beta'(0)}J(0) \in (T_pK)^{\perp}$. Also $t_0 \in (0, b]$ is said to be a *K-focal point along β* if there exists a nontrivial smooth K-Jacobi field $J \in V^{\perp}(\beta, K)$ with $J(t_0) = 0$. ■

Requiring $J \in V^{\perp}(\beta, K)$ in Definition 2.1. makes sense because it is easily checked that if J is a smooth Jacobi field along β satisfying $J(0) \in T_pK$ and $J(t_0) = 0$ for some $t_0 > 0$, then J is perpendicular to β'. Secondly, it may be checked that if $\dim M = 2$ and J is a smooth Jacobi field along β with $J(0) \in T_pK$ and $J(t_0) = 0$ for some $t_0 > 0$, then $J = 0$. Thus we have assumed $n \geq 3$ above. Particularly in codimension two (cf. [4], [7], [9]), $t_0 \in (0, b]$ is defined to be a focal point if there exists a smooth K-Jacobi field $J \in V^{\perp}(\beta)$ with $J(0) \in T_pK, J'(0) + S_{\beta'(0)}J(0) = 0$ and $J(t_0) = 0$. (In the terminology of Warner [12], such a K-Jacobi field is called a strong-K-Jacobi field in higher codimension).

The usual submanifold index form $I_{(b,K)} : V^{\perp}(\beta, K) \times V^{\perp}(\beta, K) \to \mathbf{R}$ may be defined for $X, Y \in V^{\perp}(\beta, K)$ by

$$(2.1)\qquad \begin{aligned} I_{(b,k)}(X,Y) &:= g(S_{\beta'(0)}X(0), Y(0)) \\ &\quad - \int_{t=0}^{b} (g(X',Y') - g(R(X,\beta')\beta', Y))\,dt. \end{aligned}$$

As in the null conjugate point case (cf. [4, section 9.3]), this particular index form (2.1) is not immediately useful in characterizing K-focal points. For instance, if $f : [0,b] \to \mathbf{R}$ is any smooth function with $f(0) = 0$, then $Y = f\beta' \in V^{\perp}(\beta, K)$ and $I_{(b,K)}(X,Y) = 0$ for all $X \in V^{\perp}(\beta, K)$. But such a vector field $Y = f\beta'$ is a smooth K-Jacobi field in $V_0^{\perp}(\beta, K)$ iff $f = 0$. For this reason as well as to prove the index theorem, we will work with the quotient index form $\bar{I}_{(b,K)}$ as in [7], [3], [4, chapters 9, 11]. We then need to briefly review this construction, as detailed for the conjugate point case in [4, Section 9.3].

Let

$$\begin{aligned} [\beta'(t)] &= \{\lambda\beta'(t);\ \lambda \in \mathbf{R}\} \quad \text{for any } t \in [0,b], \\ [\beta'] &= \cup\{[\beta'(t)];\quad 0 \le t \le b\}, \\ (\beta'(t))^{\perp} &= \{v \in T_{\beta(t)}M;\quad g(v, \beta'(t)) = 0\}, \\ N(\beta) &= \cup\{(\beta'(t))^{\perp};\quad 0 \le t \le b\}, \\ G(\beta(t)) &= \frac{N(\beta'(t))}{[\beta'(t)]} \end{aligned}$$

and finally the quotient bundle

$$G(\beta) = \cup\{G(\beta(t));\quad 0 \le t \le b\}.$$

We have projection maps $\pi : N(\beta(t)) \to G(\beta(t))$ given by $\pi(v) = v + [\beta'(t)]$ and $\pi : N(\beta) \to G(\beta)$ given by $\pi(X)(t) = X(t) + [\beta'(t)]$ for all $t \in [0,b]$. As in [4, section 9.3], let $\chi(\beta)$ denote the piecewise smooth sections of $G(\beta)$ and $\chi_0(\beta) = \{V \in \chi(\beta);\ V(b) = [\beta'(b)]\}$. In our context, let $\chi(\beta, K)$ denote the piecewise smooth sections $V : [0,b] \to G(\beta)$ such that $V = \pi(X)$ for some $X \in V^{\perp}(\beta, K)$ and $\chi_0(\beta, K) = \{V \in \chi(\beta, K) : V(b) = [\beta'(b)]\}$. We then have the restriction of the projection map $\pi : V^{\perp}(\beta, K) \to \chi(\beta, K)$ still given by $\pi(X) = X + [\beta']$. Note also that for $X, Y \in V^{\perp}(\beta, K)$, we have $\pi(X) = \pi(Y)$ iff $X = Y + f\beta'$ for a piecewise smooth function $f : [0,b] \to \mathbf{R}$ with $f(0) = 0$. Thus $\pi(X) = \pi(Y)$ for $X, Y \in V^{\perp}(\beta, K)$ implies $X(0) = Y(0)$.

One then obtains a positive definite quotient metric $\bar{g}$, covariant derivative operator $\bar{D}$ and curvature tensor $\bar{R}$ for $\chi(\beta, K)$ as in [7], [4 section 9.3]. Given $V, W \in$

$\chi(\beta, K)$, set $\bar{g}(V,W) = g(X,Y)$, $V'(t) = \bar{D}_{\beta'}V(t) = \pi(D_{\beta'}X(t))$ and $\bar{R}(V,\beta')\beta' = \pi(R(X,\beta')\beta')$ where $V = \pi(X)$ and $W = \pi(Y)$. Moreover, we extend the second fundamental tensor $S_{\beta'(0)}$ to the quotient bundle construction as

DEFINITION 2.2. Given $V \in \chi(\beta, K)$, define $\bar{S}_{\beta'(0)}V(0) \in \pi(T_pK)$ by $\bar{S}_{\beta'(0)}V(0) := \pi(S_{\beta'(0)}X(0)) = S_{\beta'(0)}X(0) + [\beta'(0)]$ where $V = \pi(X), X \in V^{\perp}(\beta, K)$. (Our definition of $S_{\beta'(0)}$ is well defined since if $Y \in V^{\perp}(\beta, K)$ with $\pi(X) = \pi(Y) = V$, then $X(0) = Y(0)$ as noted above). ∎

Recall that a Jacobi class V in $G(\beta)$ is a smooth vector class satisfying the differential equation $V'' + \bar{R}(V,\beta')\beta' = [\beta']$. With the above machinery in hand, we may now make the following

DEFINITION 2.3. A (smooth) Jacobi class $V \in \chi(\beta, K)$ is a *K-Jacobi class* if V satisfies the boundary conditions

(i) $V(0) \in \pi(T_pK)$,

(ii) $\bar{S}_{\beta'(0)}V(0) + V'(0) \in \pi((T_pK)^{\perp})$.

A K-Jacobi class is said to be a strong K-Jacobi class if $V(0) \in \pi(T_pK)$ and $\bar{S}_{\beta'(0)}V(0) + V'(0) = [\beta'(0)]$. ∎

It is then easy to check that V is a K-Jacobi class along β iff there exists a K-Jacobi field $J \in V^{\perp}(\beta, K)$ (as defined in Defn. 2.1) with $V = \pi(J)$. Thus these two definitions are consistent. Second, it may be shown that if $k = \dim K = n-2$, then every K-Jacobi class along β is a strong K-Jacobi class (cf. [11, Proposition 2.12]). Finally, it is also consistent with Definition 2.1 to adopt the following.

DEFINITION 2.4. $t_0 \in (0, b]$ is a *K-focal point* along β if there exists a nontrivial K-Jacobi class $V \in \chi(\beta, K)$ with $V(t_0) = [\beta'(t_0)] \in G(\beta(t_0))$. ∎

The quotient index form $\bar{I}_{(b,K)} : \chi(\beta, K) \times \chi(\beta, K) \to \mathbf{R}$ may now be defined for $V, W \in \chi(\beta, K)$ by

$$\bar{I}_{(b,K)}(V,W) := \bar{g}(\bar{S}_{\beta'(0)}V(0), W(0)) - \int_{t=0}^{b} (\bar{g}(V', W') - \bar{g}(\bar{R}(V,\beta')\beta', W))\,dt \tag{2.2}$$

and it follows that if $V = \pi(X), W = \pi(Y)$ with $X, Y \in V^{\perp}(\beta, K)$, then $\bar{I}_{(b,K)}(V,W) = I_{(b,K)}(X,Y)$.

Working along the lines of [7], [4, p. 315], Kim [11, Theorem 3.3] has established the following basic result that the quotient bundle index form characterizes K-focal points.

THEOREM 2.5. *For* $W \in \chi_0(\beta, K)$ *the following are equivalent:*

(a) W *is a* K*-Jacobi class in* $\chi_0(\beta, K)$,

(b) $\bar{I}_{(b,K)}(W, Z) = 0$ *for any* $Z \in \chi_0(\beta, K)$.

As in the conjugate point Morse theorem for null geodesic segments in [3], we want ultimately to relate the index of $\bar{I}_{(b,k)}$ to K-Jacobi fields and K-focal points along β even though the index theorem will be proven using K-Jacobi classes. Thus one needs to note the following analogue of Lemma 9.53 in [4, p. 300].

LEMMA 2.6. *Let* $W \in \chi(\beta, K)$ *be a* K*-Jacobi class with* $W(t_0) = [\beta'(t_0)]$ *for some* $t_0 > 0$. *Then there exists a unique* K*-Jacobi field* $J \in V^{\perp}(\beta, K)$ *with* $W = \pi(J)$ *and* $J(t_0) = 0$. ∎

Now let

$$J_{t_0}(\beta) = \{K\text{-Jacobi fields } J \in V^{\perp}(\beta, K);\ J(t_0) = 0\}$$

and

$$\bar{J}_{t_0}(\beta) = \{K\text{-Jacobi classes } J \in \chi(\beta, K);\ V(t_0) = [\beta'(t_0)]\}.$$

As a consequence of Lemma 2.6, we obtain

COROLLARY 2.7. *The projection map* $\pi : J_{t_0}(\beta) \to \bar{J}_{t_0}(\beta)$ *is an isomorphism.* ∎

Finally the following generalization of a well known conjugate point result (cf. Warner [13, p. 603]) is needed for the proof of the Morse Index Theorem in Section 4. Let

$$A_{\beta(t)} = \{V(t);\ V \in \chi(\beta, K) \text{ is a } K\text{-Jacobi class}\} \subset G(\beta(t))$$

and

$$B_{\beta(t)} = \{V'(t);\ V \in \chi(\beta, K) \text{ is a } K\text{-Jacobi class with } V(t) = [\beta'(t)]\} \subset G(\beta(t)).$$

Then for any $t \in [0, b]$,

$$A_{\beta(t)} \oplus B_{\beta(t)} = G(\beta(t)). \tag{2.3}$$

Especially for $t = 0$, this equation reduces to $A_{\beta(t)} = \pi(T_pK), B_{\beta(t)} = \pi((T_pK)^{\perp})$.

SECTION 3: THE FOCAL INDEX FORM

We wish to obtain a Morse index theorem in the case of an arbitrary null geodesic segment $\beta : [0,b] \to (M,g)$ which is perpendicular at $p = \beta(0)$ to a spacelike submanifold K_1 and perpendicular at $q = \beta(b)$ to a second spacelike submanifold K_2. In this setting, the relevant space of vector fields to consider will be denoted by

$$V^{\perp}(\beta, K_1, K_2) = \{Y \in V^{\perp}(\beta);\ Y(0) \in T_pK_1 \text{ and } Y(b) \in T_qK_2\}$$

and the index form $I_{(b,K_1,K_2)} : V^{\perp}(\beta, K_1, K_2) \times V^{\perp}(\beta, K_1, K_2) \to \mathbb{R}$ is given for $X, Y \in V^{\perp}(\beta, K_1, K_2)$ by

$$\begin{aligned} I_{(b,K_1,K_2)}(X,Y) := {} & g(S^1(X(0)), Y(0)) - g(S^2(X(b)), Y(b)) \\ & - \int_{t=0}^{b} (g(X', Y') - g(R(X,\beta')\beta', Y))\,\mathrm{d}t \end{aligned} \tag{3.1}$$

where $S^1 : T_pK_1 \to T_pK_1$ and $S^2 : T_qK_2 \to T_qK_2$ denote the second fundamental forms

$$S^1(v) = S_{\beta'(0)}v \text{ and } S^2(w) = S_{\beta'(b)}w$$

for $v \in T_pK_1$, $w \in T_qK_2$ associated to K_1, K_2 in the directions $\beta'(0), \beta'(b)$ respectively. We may further restrict the projection map $\pi : V^{\perp}(\beta, K_1) \to \chi(\beta, K_1)$ of the quotient bundle $G(\beta)$ defined in Section 2 and set

$$\begin{aligned} \chi(\beta, K_1, K_2) &= \pi(V^{\perp}(\beta, K_1, K_2)) \\ &= \{V \in \chi(\beta, K_1);\ V(b) \in \pi(T_qK_2)\}. \end{aligned}$$

As discussed in Section 2, the index form $I_{(b,K_1,K_2)}$ will not characterize Jacobi fields J which are K_1-Jacobi fields at $\beta(0)$ and K_2-Jacobi fields at $\beta(b)$. Thus as in the conjugate point null index theorem ([3]), we work with a quotient index form

$$\bar{I}_{(b,K_1,K_2)} : \chi(\beta, K_1, K_2) \times \chi(\beta, K_1, K_2) \to \mathbb{R}$$

given for $V, W \in \chi(\beta, K_1, K_2)$ by

$$\begin{aligned} \bar{I}_{(b,K_1,K_2)}(V,W) = {} & \bar{g}(\bar{S}^1(V(0)), W(0)) - \bar{g}(\bar{S}^2(V(b)), W(b)) \\ & - \int_{t=0}^{b} (\bar{g}(V', W') - \bar{g}(\bar{R}(V,\beta')\beta', W)\,\mathrm{d}t \end{aligned} \tag{3.2}$$

Here we use the quotient metric $\bar{g}$, covariant derivative $\bar{D}$ and curvature tensor $\bar{R}$ as defined in Section 2. Also if $X \in V^{\perp}(\beta, K_1, K_2)$ with $V = \pi(X)$, then

$$\bar{S}^1(V(0)) := S_{\beta'(0)}X(0) + [\beta'(0)] = \pi(S_{\beta'(0)}X(0))$$

and

$$\bar{S}^2(V(b)) := S_{\beta'(b)}X(b) + [\beta'(b)] = \pi(S_{\beta'(b)}X(b)).$$

Notice that the nullity of the index form (3.2) is exactly the space of K_1-Jacobi classes that are simultaneously K_2-Jacobi classes along the geodesic $\tilde{\beta}(t) = \beta(b-t)$. Also, with our choice of sign convention on the index form, $\bar{I}_{(b,K_1)}$ is negative definite on $\chi_0(\beta, K_1)$ up to the first K_1-focal point. Thus it makes sense to define

DEFINITION 3.1. The index $\bar{I}(\beta, K_1, K_2)$ of the quotient index form $\bar{I}_{(b,K_1,K_2)}$ on $\chi(\beta, K_1, K_2)$ is defined to be

$$\bar{I}(\beta, K_1, K_2) = \sup\{\dim A;\ A \subset \chi(\beta, K_1, K_2) \text{ is an } \mathbf{R}\text{-vector subspace on which } \bar{I}_{(\beta,K_1,K_2)} \text{ is positive definite}\}.$$

■

In [8, Proposition 3.7], the finiteness of $\bar{I}(\beta, K_1, K_2)$ for arbitrary β, K_1, K_2 was established using the classical piecewise smooth Jacobi field approximation technique, and the maximality of the quotient index form with respect to Jacobi classes up to the first K_1-focal point.

SECTION 4: THE MORSE INDEX THEOREM

Let (M,g) be an arbitrary Lorentzian manifold of dimension $n \geq 3$ and let $\beta : [0,b] \to (M,g)$ be a null geodesic segment which is orthogonal at $p = \beta(0)$ to a spacelike submanifold K_1 of dimension k_1 and orthogonal at $q = \beta(b)$ to a spacelike submanifold K_2 of dimension k_2. As the K_i are spacelike and $g(\beta'(0), \beta(0)) = g(\beta'(b), \beta'(b)) = 0$, we have $\beta'(0) \notin T_pK_1$ and $\beta'(b) \notin T_qK_2$. Also, $k_1, k_2 \leq n-2$. Finally we may reparametrize the null geodesic β and thus take $b = 1$. Further, in order to adapt Kalish's proof method to our quotient bundle setting, we need to assume that $t_0 = b = 1$ is not a K_1-focal point to $t = 0$, or in greater generality that $\pi(T_qK_2) \subset A_{\beta(1)}$, remembering decomposition (2.3).

Recalling that $V^{\perp}(\beta, K_1, K_2) = \{Y \in V^{\perp}(\beta);\ Y(0) \in T_pK_1 \text{ and } Y(1) \in T_qK_2\}$ and $\chi(\beta, K_1, K_2) = \{V \in \chi(\beta);\ V(0) \in \pi(T_pK_1) \text{ and } V(1) \in \pi(T_qK_2)\}$, consider the R-vector subspace $\mathcal{R}$ of $\chi(\beta, K_1, K_2)$ given by

$$\mathcal{R} = \{K_1 - \text{Jacobi classes } V \in \chi(\beta, K_1);\ V(1) \in \pi(T_qK_2)\}.$$

In view of (2.3), since $t_\sigma = 1$ is not a K_1-focal point, $\mathcal{R} \neq 0$. Also, motivated by (3.2), define the following functional $S : \mathcal{R} \times \mathcal{R} \to \mathbb{R}$ for $V, W \in \mathcal{R}$ by

$$(4.1) \qquad S(V,W) = \bar{g}(V'(1) + \bar{S}^2(V(1)), W(1)).$$

Fix $t_1, t_2, \ldots, t_k$ with $0 < t_1 < t_2 < \ldots < t_k < 1$ such that $\beta(t_1), \beta(t_2), \ldots, \beta(t_k)$ are the K_1-focal points to $t = 0$ along β. (In view of Prop. 3.7 of [8] with $K_2 = \{q\}$, there are only finitely many K_1-focal points to $t = 0$ along β). Let $m_i :=$ dim $J_{t_i}(\beta) =$ the multiplicity of the K_1-focal point $\beta(t_i)$, i.e., the dimension of the space of smooth K_1-Jacobi fields along β which vanish at t_i, (recall Corollary 2.7 here). Then the index theorem we wish to obtain may now be stated.

THEOREM 4.1. *Let $\beta : [0,1] \to (M,g)$ be a null geodesic segment in an arbitrary space-time of dimension $n \geq 3$ and let K_1, K_2 be spacelike submanifolds of dimension $\leq n-2$ which are perpendicular to β at $p = \beta(0)$ and $q = \beta(1)$ respectively. Suppose further that $t = 1$ is not a K_1-focal point to $t = 0$ along β (or that $\pi(T_q K_2) \subset A_{\beta(1)}$). Then*

$$\operatorname{Ind}(\bar{I}_{(\beta,K_1,K_2)}) = \sum_{t \in (0,1)} \dim J_t(\beta) + \operatorname{Ind}(S)$$

Here $\operatorname{Ind}(S) = \sup\{\dim \mathcal{B};\ \mathcal{B}$ *is a* $\mathbb{R}$*-vector subspace of* $\mathcal{R}$ *on which* $S|_{\mathcal{B}\times\mathcal{B}}$ *is negative definite* $\}$.

As in [3], in view of Corollary 2.7,it suffices to work with Jacobi vector classes and show that

$$\operatorname{Ind}(\bar{I}) = \sum_{t \in (0,1)} \dim \bar{J}_t(\beta) + \operatorname{Ind}(S) = \sum_{i=1}^{k} m_i + \operatorname{Ind}(S).$$

Here and in the sequel we will denote $\bar{I}_{(\beta,K_1,K_2)}$ by $\bar{I}$.

Let $\{X_1, X_2, \ldots, X_{n-2}\}$ be a basis of $(n-2)$ linearly independent K_1-Jacobi classes along β. These may be constructed as follows. Let $\eta \in (T_p K_1)^\perp$ be a null vector with $g(\eta, \beta'(0)) = -1$ and set $r = k_1 = \dim K_1$. Since g is a Lorentzian (nondegenerate) metric on $(T_p K_1)^\perp$, we may find orthonormal spacelike vectors $e_{r+1}, e_{r+2}, \ldots, e_{n-2}$ in $(T_p K_1)^\perp$ such that Span $\{e_{r+1}, \ldots, e_{n-2}\} \cap$Span $\{\eta, \beta'(0)\} = \{0\}$. Let $e_1, \ldots, e_r \in T_p K_1$ be orthonormal spacelike vectors. Then the set $\{e_1, \ldots, e_r, e_{r+1}, \ldots, e_{n-2}\}$ of spacelike vectors in $T_p M$ satisfies $g(e_i, e_j) = \delta_{i,j}, g(e_i, \beta'(0)) =$

$g(e_i, \eta) = 0$ for all i, j. Let Y_j be the unique Jacobi field along β satisfying the initial conditions $Y_j(0) = e_j, Y_j'(0) = -S_{\beta'(0)}e_j$ for $1 \le j \le r$ and satisfying the initial conditions $Y_j(0) = 0, Y_j'(0) = e_j$ for $r+1 \le j \le n-2$. Then the Y_j's are spacelike K_1-Jacobi fields along β, and if we set $X_j := \pi(Y_j) \in \chi(\beta, K_1)$, then the vector classes $\{X_1, \dots, X_{n-2}\}$ are linearly independent K_1-Jacobi classes in $\chi(\beta, K_1)$.

Consider in view of decomposition (2.3), the vector subspace $\mathcal{A}$ of $\chi(\beta, K_1, K_2)$ given by

$$\mathcal{A} = \{V \in \chi(\beta, K_1, K_2);\ V(1) = [\beta'(1)] \text{ and } V(t_i) \in A_{\beta(t_i)} \text{ for } 1 \le i \le k\}.$$

It is known that $\mathcal{A}$ may be considered as $\mathcal{A} = \{V(t) = \sum_{j=1}^{n-2} f_j(t)X_j(t);\ V(1) = [\beta'(1)]$ and $f_j : [0,1] \to \mathbb{R}$ are piecewise smooth $\}$, whence for $V \in \mathcal{A}$, one has

$$\bar{I}(V, V) = -\int_{t=0}^{1} \bar{g}\left(\sum_{j=1}^{n-2} f_j' X_j, \sum_{j=1}^{n-2} f_j' X_j\right) dt$$

which is nonpositive since the projected metric $\bar{g}$ is positive definite. Hence to calculate the index of $\bar{I}$, we wish to decompose $\chi(\beta, K_1, K_2)$ as

$$\chi(\beta, K_1, K_2) = \mathcal{A} \oplus \mathcal{A}_-^c \oplus \mathcal{A}_+^c$$

where $\mathcal{A}^c = \mathcal{A}_-^c \oplus \mathcal{A}_+^c$ is finite dimensional, $\bar{I}$ is negative semidefinite on $\mathcal{A} \oplus \mathcal{A}_-^c$, $\bar{I}$ is positive definite on $\mathcal{A}_+^c$, and

$$\dim(\mathcal{A}_+^c) = \sum_{i=1}^{k} m_i + \operatorname{Ind}(S).$$

As an intermediate step in this construction, we need to «diagonalize» $S : \mathcal{R} \times \mathcal{R} \to \mathbb{R}$ with respect to the positive definite metric $\bar{g}$ using either the hypothesis «$t = 1$ is not a K_1-focal point» or the hypothesis «$\pi(T_q K_2) \subset A_{\beta(1)}$» as follows. Choose $s = k_2 = \dim K_2$ linearly independent K_1-Jacobi classes $J_1, \dots, J_s$ in $\chi(\beta, K_1)$ so that

(i) $\{J_1(1), \dots, J_s(1)\}$ span $\pi(T_q K_2)$,

(ii) S is negative definite on $\operatorname{Span}_{\mathbb{R}}\{J_1, \dots, J_N\}$ with $N = \operatorname{Ind}(S)$ as defined in the statement of Theorem 4.1, and

(iii) S is positive semidefinite on $\operatorname{Span}_{\mathbb{R}}\{J_{N+1}, \dots, J_s\}$. (This last space includes the K_1, K_2-Jacobi classes which constitute the null space of the index form $\bar{I}$.) A basic identity in this setting for V a smooth Jacobi class along β and $W \in \chi(\beta, K_1, K_2)$ arbitrary may be written as

$$\bar{I}(V, W) = \bar{g}(V'(0) + \bar{S}^1(V(0)), W(0)) - S(V, W).$$

Hence we obtain $\bar{I} > 0$ on $\mathrm{Span}_{\mathbf{R}}\{J_1, \ldots, J_N\}$ and $\bar{I} \leq 0$ on $\mathcal{A}^c_- := \mathrm{Span}_{\mathbf{R}}\{J_{N+1}, \ldots, J_s\}$. Also standard calculations using $V(1) = [\beta'(1)]$ if $V \in \mathcal{A}$ show that $\bar{I}(\mathcal{A}, \mathcal{A}^c_-) = 0$. Thus as $\bar{I}$ is negative semidefinite on both $\mathcal{A}$ and $\mathcal{A}^c_-$, we have $\bar{I}$ negative semidefinite on $\mathcal{A} \oplus \mathcal{A}^c_-$.

It now remains to define a finite dimensional subspace $\mathcal{A}^c_+$ of $\chi(\beta, K_1, K_2)$ with $\bar{I}$ positive definite on $\mathcal{A}^c_+$ and $\chi(\beta, K_1, K_2) = \mathcal{A} \oplus \mathcal{A}^c_- \oplus \mathcal{A}^c_+$. This may be accomplished as in [10] using the decomposition $A_{\beta(t)} \oplus B_{\beta(t)} = G(\beta(t))$. Specifically, let $Y_{i,j_i} \in \chi(\beta, K_1)$ be K_1-Jacobi classes with $Y_{i,j_i}(t_i) = [\beta'(t_i)]$ for $j_i = 1, 2, \ldots, m_i = \dim \bar{J}_{t_i}(\beta)$ and $\{Y'_{i,j_i}(t_i)\}_{i=1}^{m_i}$ a $\bar{g}$-orthonormal basis for $B_{\beta(t_i)}$ for $i = 1, 2, \ldots, k$. Let Z_{i,j_i} be the smooth parallel vector class in $\chi(\beta)$ with $Z_{i,j_i}(t_i) := -Y'_{i,j_i}(t_i)$ for $i = 1, 2, \ldots, k$ and $j_i = 1, 2, \ldots, m_i$. Further construct for $i = 1, \ldots, k$ smooth bump functions $\phi_i : [0, 1] \to [0, 1]$ with mutually disjoint support such that $\phi'_i(t_i) = 0$, $\phi_i(t_i) = 1$, as in Figure 1 of [10, p. 344]. Then define vector classes $V_{i,j_i} \in \chi(\beta, K_1)$ for a positive parameter $\lambda > 0$ to be chosen later to make the index positive definite on the span of the V'_{i,j_i}'s as follows:

$$V_{i,j_i}(t) = \begin{cases} Y_{i,j_i}(t) + \lambda\phi_i(t) Z_{i,j_i}(t) & \text{if } 0 \leq t \leq t_i \\ \lambda\phi_i(t) Z_{i,j_i}(t) & \text{if } t_i \leq t \leq 1. \end{cases}$$

Now let $\mathcal{A}^c_+ \subset \chi(\beta, K_1)$ be given by

$$\mathcal{A}^c_+ := \mathrm{Span}_{\mathbf{R}}\{V_{i,j_i}, J_q; i = 1, 2, \ldots, k, j_i = 1, 2, \ldots, m_i, q = \\ = 1, 2, \ldots, N\}$$

Just as in [10], one checks readily using decomposition (2.3) that $\dim \mathcal{A}^c_+ = \sum_{i=1}^{k} m_i + \mathrm{Ind}(S)$.

Standard calculations show that

$$\bar{I}\left(\sum_i \alpha_{i,j_i} V_{i,j_i}, \sum_k \beta_k J_k\right) = 0$$

where the J_k's are the K_1-Jacobi classes diagonalizing S chosen above. Since the ϕ_i's and Z_{i,j_i}'s are given, one may check as in [10, p. 347] that for some $\lambda > 0$ sufficiently small, $\bar{I}$ is positive definite on $\mathrm{Span}_{\mathbf{R}}\{V_{i,j_i}\}$. Hence, $\bar{I}$ is positive definite on $\mathcal{A}^c_+$.

Now set $\mathcal{A}^c := \mathcal{A}^c_- \oplus \mathcal{A}^c_+$. As in [10, p. 544], it is easy to check that $\mathcal{A} \cap \mathcal{A}^c = \{0\}$. We need to show that $\mathcal{A} \oplus \mathcal{A}^c = \chi(\beta, K_1, K_2)$. Thus suppose $W \in \chi(\beta, K_1, K_2)$ is arbitrary. First, write $W(1) = \sum_{j=1}^{s} \lambda_j J_j(1)$ for $\lambda_j \in \mathbf{R}$. Then at each K_1-focal point t_i, write $W(t_i) = x_i + y_i \in G(\beta(t_i))$ with $x_i \in B_{\beta(t_i)}$ and $y_i \in B_{\beta(t_i)}$. Expand $y_i = \sum_{j_i=1}^{m_i} \lambda_{ij_i} V_{ij_i}(t_i)$ using the fact that $\{Z_{ij_i}(t_i)\}$ forms a basis for $B_{\beta(t_i)}$. Then $U := \sum_{j=1}^{s} \lambda_j J_j + \sum_{t=1}^{k} \sum_{j_i=1}^{m_i} \lambda_{ij_i} V_{ij_i}$ is an element of $\mathcal{A}^c$. Also, $V := W - U$ has the properties that $V(1) = [\beta'(1)]$ and $V(t_i) \in A_{\beta(t_i)}$ for $1 \le i \le k$. Hence $V \in \mathcal{A}$ and $W = V + U \in \mathcal{A} \oplus \mathcal{A}^c$ as required.

Finally we have $\chi(\beta, K_1, K_2) = \mathcal{A} \oplus \mathcal{A}^c = (\mathcal{A} \oplus \mathcal{A}^c_-) \oplus \mathcal{A}^c_+$ with $\bar{I}$ restricted to $(\mathcal{A} \oplus \mathcal{A}^c_-) \times (\mathcal{A} \oplus \mathcal{A}^c_-)$ negative semidefinite, $\bar{I}$ restricted to $\mathcal{A}^c_+ \times \mathcal{A}^c_+$ positive definite, and $\dim(\mathcal{A}^c_+) = \sum_{i=1}^{k} m_i + \operatorname{Ind}(S)$. It follows that $\operatorname{Ind}(\bar{I}) = \dim(\mathcal{A}^c_+) = \sum_{i=1}^{k} m_i + \operatorname{Ind}(S)$ as required. ∎

Vector fields defined like the vector classes V_{ij_i} also play an important role in the development of the traditional Morse theory, cf. [4, pp. 256-257] for a standard example.

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